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CONTROL OF PLANTS WHOSE REPRESENTATION CONTAINS DERIVATIVES OF THE CONTROL VARIABLE

by W. W. Schmaedeke

Prepared under Contract No. NASw-563 *by*
MINNEAPOLIS-HONEYWELL REGULATOR COMPANY
Minneapolis, Minnesota
for

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CONTROL OF SYSTEMS CONTAINING DERIVATIVES OF THE CONTROL VARIABLE*

by Wayne Schmaedeke‡

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ABSTRACT

Problems of control of plants modeled by differential equations containing derivatives of the control (or forcing) functions are discussed. These right side plant dynamics in conjunction with relay type jumps in the forcing functions require an analysis or synthesis method to accommodate derivatives of step functions. Previously this problem has been avoided formally by transforming to a special set of coordinates in which only zero order forcing terms appear. This paper develops a mathematical model for equations with derivatives of the control functions and establishes conditions under which the formal transformation referred to above can be rigorously applied.

Author

MATHEMATICAL MODEL OF A CONTROL SYSTEM CONTAINING DERIVATIVES OF THE CONTROL VARIABLES

In the following, $\mathcal{D}$ will be a domain in the (t,x)-space, f(t,x) will be an n-vector function defined on $\mathcal{D}$ and u(t) will be an r-vector each of whose components are of bounded variation and continuous from the right on an interval I_1. The function

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$g(t)$ will be a continuous nxr matrix defined on I_1 and (t_o,x_o) will be a point in $\mathcal{D}$ with t_o also in I_1.

In a control problem, one is given a differential equation

$$\frac{dx}{dt} = f(t,x,u) + g(t)\frac{du}{dt} \tag{$\mathcal{S}$}$$

involving f, g, u, and x. The operations of differentiation are to be understood in the sense of distribution derivatives and the equation will be called a measure differential equation because the distribution derivative of a function of bounded variation can always be identified with a measure. The problem of control is to select the r-vector $u(t)$ on an interval of time $[t_o,t_1]$ so that the solution (response) of ($\mathcal{S}$) will initiate at a prescribed point x_o at the time t_o and behave in a prescribed manner on the interval $[t_o,t_1]$. For example, it may be desired to steer the response from x_o to a continuously moving target set $G(t)\subset R^n$.

It will be convenient to change the notation in ($\mathcal{S}$) to conform with the previous remarks and to write the equation as

$$Dx = f(t,x,u) + g(t)Du, \quad x(t_o) = x_o. \tag{$\mathcal{M}$}$$

Here the notation Dx means the distribution derivative of the function x.

<u>DEFINITION 1</u>. A solution $x(t)$ of ($\mathcal{M}$) is a real bounded variation n-vector $x(t)$ together with an interval I containing the given initial time t_o such that $x(t)$ is continuous from the right on I and

(i) $(t,x(t)) \in \mathcal{D}$ for $t \in I$

(ii) $x(t_o) = x_o$

(iii) the distribution derivative of $x(t)$ on I is $f(t,x) + g(t)Du$.

Next, the integral equation

$$x(t) = x_o + \int_{t_o}^{t} f(s,x(s),u(s))ds + \int_{t_o}^{t} g(s)du(s). \qquad (\mathcal{I})$$

will be considered.

<u>DEFINITION 2</u>. A solution $x(t)$ of $(\mathcal{I})$ is a real bounded variation n-vector $x(t)$ together with an interval I such that

(i) $(t,x(t)) \in \mathcal{D}$ for $t \in I$

(ii) $x(t)$ satisfies the integral equation

<u>REMARK</u>: A solution $x(t)$ of $(\mathcal{I})$ is necessarily continuous from the right. Also, $x(t)$ has discontinuities where u does.

<u>THEOREM 1</u>. A solution $x(t)$ of $(\mathcal{I})$ is a solution $x(t)$ of $(\mathcal{M})$ and conversely.

A proof of Theorem 1 is given in reference 1. Also included in that reference are a number of theorems relating to existence and uniqueness of solutions to $(\mathcal{M})$, both locally and globally.

For convenience in discussing the control problem for an equation of the form $(\mathcal{M})$, the following assumptions regarding the coefficients are made:

(i) $f(t,x,u)$, $\frac{\partial f}{\partial x}(t,x,u))$ are real continuous functions in $R^1 \times R^n \times \Omega$ where R^n is the real n-dimensional number space and Ω is a non-empty compact subset of R^n.

(ii) $g(t)$ is a continuous nxr-matrix on R^1

(iii) The functions $u(t)$ are of bounded variation and continuous from the right on appropriate time intervals such that the graph of $u(t)$ lies in Ω.

For each function $u(t)$ as in (iii), but defined on a finite interval $[t_o,t_1]$, the measure differential equation $(\mathcal{M})$ has a unique bounded variation solution (called a response) on $[t_o,t_1]$ (or a subinterval) through a prescribed initial point (t_o,x_o). This is a result of the existence and uniqueness theorems of Chapter I of reference 1. The representation of a response is, of course, the unique bounded variation solution of the integral equation

$$x(t) = x_o + \int_{t_o}^{t} f(s,x(s),u(s))ds + \int_{t_o}^{t} g(s)du(s). \qquad (\mathcal{Q})$$

<u>DEFINITION 3</u>. A control for the system $(\mathcal{M})$ where a non-empty compact set Ω contained in R^r and an initial point x_o in R^n have been prescribed, is a vector valued function $u(t)$ of bounded variation and continuous from the right on a finite interval $[t_o,t_1]$ with its graph in Ω such that its response $x(t)$ with $x(t_o) = x_o$ is also defined in R^n on $[t_o,t_1]$.

<u>DEFINITION 4</u>. For a given real valued continuous function $f^o(t,x,u)$ defined on $R^1 \times R^n \times \Omega$, the cost functional $C(u)$ of a control $u(t)$ on $[t_o,t_1]$ with response $x(t)$ is defined by

$$C(u) = \int_{t_o}^{t_1} f^o(s,x(s),u(s))ds.$$

If $f^o(t,x,u) \equiv 1$ then $C(u) = t_1 - t_o$, the time duration over which the control is exerted.

<u>DEFINITION 5</u>. Given the control problem for

a) $Dx = f(t,x,u) + g(t)Du$ with f, g, and u as previously described; the following data is also given:

b) a non-empty compact restraint set $\Omega \subset R^r$ (containing the graphs on the controls)

c) $x_o \in R^n$, the initial point

d) $G(t) \subset R^n$ on $[\tau_o, \tau_1]$ the continuously moving target set

e) $C(u) = \int_{t_o}^{t_1} f^o(s,x(s),u(s))ds$, the cost functional.

For a given number $E > 0$, the set $\Delta = \Delta\,(f(t,x,u),\ g(t),\ \Omega,\ x_o,\ G(t),\ E)$ is defined as the set of all controls u(t) in Ω with u(t) of bounded variation and right continuous on subintervals $[t_o,t_1] \subset [\tau_o,\tau_1]$ such that the total variation of each function u(t) on its interval $[t_o,t_1]$ is less than or equal to E, and with responses x(t) such that $x(t_o) = x_o$ and $x(t_1) \in G(t_1)$. This set Δ is called the set of admissible controls.

<u>DEFINITION 6</u>. A control $u^*(t)$ in Δ is called optimal in case

$$C(u^*) \leq C(u)$$

for every u(t) in Δ.

<u>REMARK:</u> The hypothesis concerning the uniform bounded total variation of the admissible controls is concerned with the fact that in a large class of problems the total variation is a mathematical manifestation of the motion of some process. It is those processes which contain devices capable of sustaining only a bounded amount of movement, regardless of the admissible control that is applied, to which the following theorem pertains.

THEOREM 2. Given the control problem described in Definition 5 with the further restriction that there exist a non-decreasing function $h(t)$ continuous from the right such that all $u(t)$ in Δ satisfy the inequalities

$$|u(\beta) - u(\alpha)| \leq h(\beta) - h(\alpha)$$

for every subinterval $[\alpha,\beta]$ of the interval $[t_o, t_o + \delta]$ for some appropriate $\delta > 0$, however small.

It will be assumed that the set Δ is such that

A) Δ is not empty

B) There exists a real bound $B < \infty$ such that for all responses $x(t)$ corresponding to controls in Δ we have $|x(t)| \leq B$.

CONCLUSION: Then there exists an optimal control in Δ.

REMARK: It is assumed that x_o is not in the target $G(t_o)$, then the inequalities between u and h guarantee that $x(t)$ lies outside $G(t)$ for all responses and all t sufficiently near t_o. If the functions $u(t)$ in Δ satisfy a uniform Lipschitz condition for all t in a neighborhood of t_o then the function $h(t)$ may be taken to be a multiple of t.

The proof of Theorem 2 may be found in reference 1.

A STUDY OF A CERTAIN LINEAR EQUATION

The linear ordinary differential equation

$$\ddot{x} + a_1\dot{x} + a_ox = b_1\dot{u} + b_2u. \qquad (\mathcal{E})$$

will be considered. This equation might be obtained, for example, in the analysis of the control of aircraft. (Reference 2, where the equations of longitudinal-symmetric motion of a rigid aircraft are derived, contains the details). In particular, the small amplitude pitch motions of many aircraft can be summarized apprcximately by solutions of the equations

$$\dot{\alpha} - k_1\alpha - \dot{\theta} = k_2\delta \text{ (Flight Path equation)} \tag{1}$$

$$\ddot{\theta} - k_3\dot{\theta} - k_4\alpha - k_5\dot{\alpha} = k_6\delta \quad \text{(Pitch equation)} \tag{2}$$

Here the k_j are constants while α corresponds to the aircraft fuselage reference angle of attack, θ corresponds to pitch attitude change, and δ denotes elevator deflection. It will be convenient to define $x = \dot{\theta}$ and to eliminate α and $\dot{\alpha}$ from (1), (2) and the equation obtained from differentiating (2). The result is

$$\ddot{x} - (k_1+k_3+k_5)\dot{x} + (k_1k_3-k_4)x = (k_6+k_2k_5)\dot{\delta} + (k_2k_4-k_1k_6)\delta \tag{3}$$

The equation (3) is precisely of the form (ξ). It is a problem of considerable importance in engineering to treat this equation by allowing the control variable $u(t)$ to be of the relay type, i.e., to have discontinuities of the first kind. The presence of a derivative of $u(t)$ prevents existing theories from being applied and the problem is avoided formally by transforming to a set of coordinates in which the derivatives of the control variable do not appear, (reference 3, p. 191 supplies the details).

To be more specific now, the equation (ξ) will be considered where a_1, a_o, b_1, b_2 are constants and where $x(t_o) = x_o$, $\dot{x}(t_o) = C_o$, $|u(t)| \leq \alpha$ for $\alpha > 0$, and a_1, $a_o > 0$, $b_1 \neq 0$, $b_2 \neq 0$. The target will be any compact set containing the origin of the $(x,\dot{x})$-plane and the cost functional will be given as

$$C(u) = \int_{t_o}^{t_1} f^o(x(s),\dot{x}(s), u(s))ds.$$

At this point (ξ) will be reduced to a linear system free from derivatives of the control variable, (reference 3 p. 191 supplies the details). To this end, x is defined by

$$x = \hat{x}_1 + G_o(t)u$$

for as yet undetermined, $G_o(t)$ and $\hat{x}_2$ is defined by

$$\dot{\hat{x}}_1 = \hat{x}_2 + G_1(t)u$$

for undetermined $G_1(t)$. Then it is clear that $\dot{\hat{x}}_2$ is given by

$$\dot{\hat{x}}_2 = -a_o\hat{x}_1 - a_1\hat{x}_2 + G_2(t)u$$

where G_o, G_1, and G_2 are determined by eliminating $\hat{x}_1$ and $\hat{x}_2$ from the above and requiring that the resultant differential equation agree with (ξ). The result is

$$G_o(t) = 0$$
$$G_1(t) = b_1$$
$$G_2(t) = b_2 - a_1b_1.$$

The system

$$\dot{\hat{x}}_1 = \hat{x}_2 + b_1u$$
$$\dot{\hat{x}}_2 = -a_o\hat{x}_1 - a_1\hat{x}_2 + (b_2 - a_1b_1)u \qquad (\theta)$$

is obtained with initial conditions

$$\hat{x}_1(t_o) = x_o$$
$$\hat{x}_2(t_o) = C_o - b_1u(t_o).$$

It is observed that the initial point (x_o, C_o) has been transformed to a line segment in the $(\hat{x}_1, \hat{x}_2)$-plane and that the target G would be similarly "enlarged". For simplicity, it is assumed that the origin of the $(\hat{x}_1, \hat{x}_2)$-plane is in the interior of $G(\hat{x}_1,\hat{x}_2)$. The cost functional becomes

$$C_o(u) = \int_{t_o}^{t_1} f^o(\hat{x}_1(s), \hat{x}_2(s) + b_1u(s), u(s))ds.$$

The ordinary system ($\mathcal{O}$) does not contain derivatives of the control variables and can therefore be studied using the conventional theory.

Next, ($\mathcal{E}$) is written as a linear system by making no attempt to eliminate the derivatives of the control variable, i.e., by proceeding in a natural way by defining

$$x_1 = x$$

the following system is obtained:

$$\begin{aligned} \mathring{x}_1 &= x_2 \\ \mathring{x}_2 &= -a_o x_1 - a_1 x_2 + b_1 \mathring{u} + b_2 u. \end{aligned} \qquad (\mathcal{E}')$$

At this point, the class of admissible controls is taken to be the functions u(t) of uniform bounded variation, with $v(u,[t_o, t_1]) \leq M$, such that $|u| \leq \alpha$.[1] Then ($\mathcal{E}'$) is written as a measure differential system

$$\begin{aligned} Dx_1 &= x_2 \\ Dx_2 &= -a_o x_1 - a_1 x_2 + b_1 Du + b_2 u. \end{aligned} \qquad (\mathcal{M})$$

The initial conditions in this instance are

$$x_1(t_o) = x_o$$

$$x_2(t_o) = C_o$$

and the cost functional is

---

[1] $v(u,[t_o,t_1])$ means the total variation of the vector u(t) on the interval $[t_o,t_1]$, i.e., each component of u(t) is of uniform bounded total variation.

$$C_{\mathcal{M}}(u) = \int_{t_o}^{t_1} f^o(x_1(s),\ x_2(s),\ u(s))ds.$$

The target is assumed to have been defined by $G(x, \dot{x}) \leq 0$ in which case the target becomes the set in the (x_1,x_2)-plane defined by $G(x_1,x_2) \leq 0$.

It is shown in reference 1 that the set $\Delta_{\mathcal{M}}$ for this problem is not empty.

Now the responses to a linear system are easily shown to be uniformly bounded and the set Δ of admissible controls with variations limited by some vary large constant E which transfer $X_o = (x_1(t_o)\ x_2(t_o))$ to the origin is clearly not empty since the domain of controllability is the entire $(x_1,\ x_2)$-plane. Hence there exists an optimal control $u^*(t)$ in Δ which transfers the point $X_o = (x_o,C_o)$ to the target $G(x_1,x_2) \leq 0$ because the target was assumed to contain the origin of R^2. The question resolved below is under what conditions will $u^*(t)$ also be the optimal control for the system $(\mathcal{O})$ when that system is studied using the conventional methods?

A connection is first established between the components of the two vector solutions $X(t) = (x_1(t),\ x_2(t))'$ and $\hat{X}(t) = (\hat{x}_1(t),\ \hat{x}_2(t))'$ of $(\mathcal{M})$ and $(\mathcal{O})$ respectively.

<u>THEOREM 3</u>. If $u(t)$ is any function of bounded variation, then the first components of $X(t)$ and $\hat{X}(t)$ are identical and the second components are related by

$$x_2(t) - \hat{x}_2(t) = b_1u(t).$$

<u>PROOF</u>: It is observed that the systems $(\mathcal{O})$ and $(\mathcal{M})$ have the same homogeneous part and that the fundamental solution matrix

is e^{At} where

$$A = \begin{bmatrix} 0 & 1 \\ -a_o & -a_1 \end{bmatrix} .$$

Hence, forming the difference $X(t) - \hat{X}(t)$ by using the variation of parameters representations for the solutions of each, there results

$$X(t) - \hat{X}(t) = e^{A(t-t_o)}(X_, -\hat{X}_o) + \int_{t_o}^{t} e^{A(t-s)} \begin{bmatrix} -b_1 \\ a_1 b_1 \end{bmatrix} u(s)ds +$$

$$+ \int_{t_o}^{t} e^{(t-s)A} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} du(s).$$

Now, by integration by parts:

$$\int_{t_o}^{t} e^{A(t-s)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} du(s) = - \int_{t_o}^{t} \frac{d}{ds} e^{A(t-s)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u(s)ds +$$

$$+ \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u(t) - e^{A(t-t_o)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u_o .$$

Thus

$$X(t) - \hat{X}(t) = e^{A(t-t_o)} [X_o - \hat{X}_o - \begin{bmatrix} 0 \\ b_1 u_o \end{bmatrix}] + \begin{bmatrix} 0 \\ b_1 u(t) \end{bmatrix} +$$

$$+ \int_{t_o}^{t} e^{A(t-s)} \{ \begin{bmatrix} 0 \\ b_2 \end{bmatrix} + A \begin{bmatrix} 0 \\ b_1 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 - a_1 b_1 \end{bmatrix} u(s)ds,$$

but $A \begin{bmatrix} 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ -a_1 b_1 \end{bmatrix}$ and

$$X_o - \hat{X}_o - \begin{bmatrix} 0 \\ b_1 u_1 \end{bmatrix} = \begin{bmatrix} x_o \\ C_o \end{bmatrix} - \begin{bmatrix} x_o \\ C_o - b_1 u_o \end{bmatrix} - \begin{bmatrix} 0 \\ b_1 u_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the above reduced to

$$X(t) - \hat{X}(t) = \begin{bmatrix} 0 \\ b_1 u(t) \end{bmatrix} . \quad \text{QED.}$$

A comparison of the cost functionals $C_{\mathcal{O}}(u)$ and $C_{\mathcal{M}}(u)$ when u is a function of bounded variation shows by virtue of Theorem 3 that they are identical.

Returning now to the study of system $(\mathcal{O})$, $\Delta_{\mathcal{O}}$ is defined to be the set of all measurable functions u(t) whose graphs lie in the set Ω: $\{u: |u| \leq \alpha\}$ and whose responses originate on the closed interval $\hat{x}_1(t_o) = x_o$, $C_o - \frac{\alpha}{2} \leq \hat{x}_2(t_o) \leq C_o + \frac{\alpha}{2}$, and which are transferred to the target at time t_1. It is shown in reference 1 that $\Delta_{\mathcal{O}}$ is not empty.

It is to be noticed that if u*(t) is substituted in $C_{\mathcal{O}}(u)$, there results

$$C_{\mathcal{O}}(u^*) \geq C_{\mathcal{O}}(u).$$

Next, since the system $\mathcal{O}$) is proper (i.e., rank [B, AB] = 2) then the optimal control u(t) will be relay if, for example, u appears linearly in $f^o(x,t)$, the integrand of the cost functional. Being a relay or step function (assuming values $+\alpha$ or $-\alpha$ on finite segments of time) the optimal control u is of bounded variation. Thus the two cost functionals $C_{\mathcal{O}}(u)$ and $C_{\mathcal{M}}(u)$ are identical and there results (by definition of u* as minimizing $C_{\mathcal{M}}(u)$ over all bounded variation controls whose total variation is uniformly bounded)

$$C_{\mathcal{M}}(u^*) \leq C_{\mathcal{M}}(\hat{u}).$$

But also the inequalities

$$C_{\mathcal{M}}(\hat{u}) = C_{\mathcal{O}}(\hat{u}) \leq C_{\mathcal{O}}(u^*) = C_{\mathcal{M}}(u^*) \leq C_{\mathcal{M}}(\hat{u})$$

imply $C_{\mathcal{O}}(\hat{u}) = C_{\mathcal{O}}(u^*)$ and thus $\hat{u}(t)$ is an optimal control for the system $(\mathcal{M})$ because it has the desired response. Also when transforming back to x and $\dot{x}$ in $(\mathcal{E})$ there results

from $(\mathcal{M})$:

$$x(t) = x_1(t)$$
$$\dot{x}(t) = x_2(t)$$

from $(\mathcal{O})$:

$$x(t) = \hat{x}_1(t)$$
$$\dot{x}(t) = \hat{x}_2(t) + b_1 u(t).$$

But by Theorem 3, these are identical sets of data.

CONCLUSION

The optimal control problem for

$$\ddot{x} + a_1\dot{x} + a_o x = b_1\dot{u} + b_2 u \qquad (\mathcal{E})$$

can be treated either by the system

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + b_1 u \\ \dot{\hat{x}}_2 &= -a_o\hat{x}_1 - a_1\hat{x}_2 + (b_2 - a_1 b_1)u \end{aligned} \qquad (\mathcal{O})$$

or by the measure system

$$Dx_1 = x_2 \qquad (\mathcal{M})$$

$$Dx_2 = -a_o x_1 - a_1 x_2 + b_1 Du + b_2 u.$$

The system $(\mathcal{O})$ involves an enlargement of the initial state and target and leads to confusion. However, the theory of optimal control developed above for measure systems $(\mathcal{M})$ enables a straight forward treatment of the optimal control to be given which agrees with $(\mathcal{O})$.

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